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Comparison of Finite Volume Canonical and Grand Canonical Gibbs Measures: The Continuous Case

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We consider a continuous gas with finite range positive pair potential and we assume that the cluster expansion convergence condition holds. We prove a sharp bound on the difference between the finite volume grand canonical and canonical expectation of local observable. The bound is given in terms of the support of the observable, of its grand canonical variance and of the volume on which the system is confined.

KEY WORDS: Continuous systems; Gibbs measures; equivalence of ensembles.

1. INTRODUCTION

The equivalence of ensembles is one of the central problems of statistical mechanics and traces back to Gibbs (1902). As far as the thermodynamic functions is concerned under suitable conditions on the interaction, this question is already well understood.[(1,2)] The equivalence of ensembles as been studied also at the level of measures and important results have been obtained. Classical results state that the difference between the canonical and grand canonical expectation of a local observable vanishes when the volume goes to infinity and the support of the observable is kept fixed (see e.g., refs. 3–5 and references therein). Recently the possibility to obtain sharper estimates has been widely investigated and the main motivations come from:

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(i) the theory of stochastic spin exchange dynamics reversible w.r.t. the canonical Gibbs measure of finite volume,[6,7]

(ii) the theory of Renormalization group pathologies[8]

(iii) the theory of random matrices.[9]

In order to improve over classical results different methods have been envisaged, mostly for lattice compact spins models (see e.g., refs. 7, 8 and 10). In particular, in ref. 10 the case of a general lattice discrete spin model satisfying a suitable mixing property has been analyzed and optimal estimates has been established. Contrary to the methods developed in ref. 7 or 8, the techniques of ref. 10 completely avoid proving a local central limit theorem and pose negligible restrictions on the size of the support of the observable. In the present paper, motivated by a rigorous analysis of the so-called *Boltzmann–Gibbs principle* for the equilibrium fluctuations of interacting Brownian and Ornstein-Uhlenbeck particles processes,[11,12] we extend the approach of ref. 10 to a continuous system of particles interacting through a finite range positive pair potential. In particular we prove that, under a suitable smallness condition on the activity z (see condition (CE) before Theorem 2.1),

$$\left|\nu(f)-\mu(f)\right|\leqslant C\,\mu(f,f)^{1/2}\,\frac{\max\{z|\Delta|,\sqrt{z|\Delta|}\}}{z|\Lambda|},\tag{1.1}$$

where ν and μ are, respectively, the canonical and grand canonical Gibbs measure in the region Λ, z is the activity and it is such that the mean grand canonical number of particles coincides with the (fixed) canonical value, $|\Delta|$ is the support of the observable f and C is a positive constant independent of f.

In the case of the continuous gas, the main difficulty comes from the fact that the number of particles that can be contained in any fixed and finite volume is not bounded. This problem is essentially bypassed assuming the (CE) condition. This condition plays an important role not only because it assures a strong mixing property (decay of correlations) of the grand canonical measure, crucial in the proof of (1.1), but also because it gives a tight control on the large deviations of the local number of particles.

Our result improves over the one obtained by Spohn in 1986. Indeed, in Lemma 13 in ref. 13, it is proven that, if z satisfies (CE) and if $\hat{\mu}$ denotes the unique infinite volume Gibbs measure, then

$$\lim_{\Lambda\nearrow\mathbb{R}^d}|\Lambda|\;\hat{\mu}\left[(\nu(f)-\mu(f))^2\right]=0,\tag{1.2}$$

where f is a C^∞-function with compact support, which depends on the number of particles on finite regions and such that $\hat{\mu}(f)=0$.

The paper is organized as follows. In Section 2, we introduce the notations and give the main theorem. In Section 3, we prove the theorem using the technical results contained in Section 4.

2. NOTATIONS AND RESULTS

Let $\mathcal{B}(\mathbb{R}^d)$ the collection of finite (measurable) subsets of $\mathbb{R}^d$. For $A\in\mathcal{B}(\mathbb{R}^d)$, we denote by $|A|$ the Lebesgue measure of A. The configuration space is the set Ω of all locally finite subsets of $\mathbb{R}^d$:

$$\Omega \doteq \{\ \omega\subset\mathbb{R}^d : \mathrm{card}(\omega\cap A)<\infty\ \ \forall A\in\mathcal{B}(\mathbb{R}^d)\ \},$$

where $\mathrm{card}(A)$ stands for the cardinality of A. We define the counting variables $N_A:\omega\to\mathrm{card}(\omega\cap A)$, where $A\in\mathcal{B}(\mathbb{R}^d)$. Given $\eta,\omega\in\Omega$, we let $\eta\triangle\omega$ be the symmetric difference of η and ω, i.e., $\eta\triangle\omega\doteq(\eta\cup\omega)\setminus(\eta\cap\omega)$. For $\Lambda\in\mathcal{B}(\mathbb{R}^d)$, we consider the finite volume configuration space

$$\Omega_\Lambda\doteq\{\omega\subset\Lambda:\omega \text{ is finite}\}$$

A function f is called a local function if there exists a set $A\in\mathcal{B}(\mathbb{R}^d)$ such that f depends only on the configuration inside A, i.e., on $\omega\cap A$, and A will be its support.

For $x,y\in\mathbb{R}^d$, the Euclid distance is denoted by $d(x,y)$ and we write $|x|$ for $d(x,0)$. Finally, by Q_l we denote the cube of all $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$ such that $x_i\in[0,l]$. If $x\in\mathbb{R}^d$, $Q_l(x)$ stands for Q_l+x.

**Regular sets:* a finite subset Λ of $\mathbb{R}^d$ is said to be a *l–regular*, $l\in\mathbb{R}_+$, if there exists $x\in\mathbb{R}^d$ such that Λ is the union of a finite number of cubes $Q_l(x^i+x)$ where $x^i\in l\mathbb{Z}^d$. This means that there exists a set of indexes I_Λ such that $\Lambda=\cup_{i\in I_\Lambda}Q_l(x^i+x)$. We denote the class of all such sets by $\mathbb{F}_l$. The l-support of a function f of support A is the smallest l-regular set Δ such that $A\subset\Delta$. Given a set $\Lambda\in\mathbb{F}_l$, we define $\partial_r^-\Lambda\doteq\{x\in\Lambda\mid d(x,\Lambda^c)\leqslant r\}$ for some positive real r.

**The Hamiltonian:* Let $\phi:\mathbb{R}^d\longrightarrow\mathbb{R}$ be a measurable function. We assume the following on the pair potential ϕ:

(1) ϕ is an even function on $\mathbb{R}^d$ and it has a finite range: take $R>0$ such that $\phi(x)=0$ if $|x|>R$.

(2) ϕ is positive.

For Λ a finite measurable subset of $\mathbb{R}^d$, the Hamiltonian $H_\Lambda : \Omega \longrightarrow \mathbb{R}$ is given by

$$H_\Lambda(\omega) = \sum_{\substack{\{x,y\}\subset\omega \\ \{x,y\}\cap\Lambda\neq\emptyset}} \phi(x-y)\,.$$

For ω and η in Ω, we define $H_\Lambda^\eta(\omega) \doteq H_\Lambda(\omega_\Lambda \eta_{\Lambda^c})$, where $\omega_\Lambda \eta_{\Lambda^c} = (\omega \cap \Lambda) \cup (\eta \cap \Lambda^c)$. Λ^c is the complement of Λ and η is called the boundary condition.

**The Gibbs measures:* We denote by $\mu_{z,\Lambda}^\eta(f)$ the expectation of f w.r.t. the grand canonical Gibbs measure $\mu_{z,\Lambda}^\eta$ with activity z, boundary condition η, volume Λ, while $\mu_{z,\Lambda}(f)$ denotes the function $\omega \to \mu_{z,\Lambda}^\omega(f)$. Explicitly, for all measurable functions f on Ω_Λ, we have

$$\mu_{z,\Lambda}^\eta(f) = \frac{1}{Z_{z,\Lambda}^\eta} \sum_{k=0}^{+\infty} \frac{z^k}{k!} \int_{\Lambda^k} e^{-\beta H_\Lambda^\eta(x)} f(x)\ dx,$$

where we have identified the functions on Ω_Λ with the symmetric functions on $\cup_{n=0}^\infty \Lambda^n$, $Z_{z,\Lambda}^\eta$ is the appropriate normalization factor. Moreover, we write $\mu_{z,\Lambda}^\eta(f,g)$ to denote the covariance of f and g w.r.t. the measure $\mu_{z,\Lambda}^\eta$ (when it exists). We denote by $\nu_{\Lambda,N}^\eta(f) := \mu_{z,\Lambda}^\eta(\ f \mid N_\Lambda = N)$ the expectation of f w.r.t. the canonical Gibbs measure with N particles, on the volume Λ, and with η as boundary condition. Explicitly

$$\nu_{\Lambda,N}^\eta(f) = \frac{1}{Z_{\Lambda,N}^\eta} \int_{\Lambda^N} e^{-\beta H_\Lambda^\eta(x)} f(x)\ dx,$$

where $Z_{\Lambda,N}^\eta$ is the appropriate normalization factor.
We omit for simplicity here and in almost all the paper the dependence on β.

For a subset $X \in \Omega$, we set $\mu_{z,\Lambda}(X) = \mu_{z,\Lambda}(\mathbf{1}_X)$, where $\mathbf{1}_X$ is the indicator function on X. The grand canonical Gibbs measure satisfies the DLR compatibility conditions

$$\mu_{z,\Lambda}^\eta(\mu_{z,V}(X)) = \mu_{z,\Lambda}^\eta(X) \quad \forall X \in \mathcal{F}\ \forall V, \Lambda \in \mathcal{B}(\mathbb{R}^d),\ \ V \subset \Lambda. \qquad (2.1)$$

**Cluster Expansion and Strong Mixing Condition:* In order to prove our main result, we need some kind of mixing property of the grand canonical Gibbs measure, which can be proved under the hypothesis of a convergent cluster expansion. An explicit condition which guarantees this convergence is the following: let $\hat{z}_0(\beta,\phi) \doteq \left(e \int_{\mathbb{R}^d} (1-e^{-\beta\phi(q)})\ dq\right)^{-1}$. Then assume that

$$0 < z \leqslant z_0 < \frac{1}{3}\,\hat{z}_0(\beta,\phi) \tag{CE}$$

Under hypothesis of positive interaction and (CE), there exists a unique grand canonical Gibbs measure (see Ref. 1).

Here is our main theorem.

Theorem 2.1. Assume (CE). Let $\delta > 0$ and N be a possible value of the number of particles. For a fixed $\Lambda \in \mathbb{F}_{2R+\delta}$, we assume that, given a boundary condition $\eta \in \Omega$, the grand canonical Gibbs measure is such that $\mu^{\eta}_{z,\Lambda}(N_\Lambda) = N$ and set $\nu^{\eta}_{\Lambda,N}(\cdot) := \mu^{\eta}_{z,\Lambda}(\cdot \mid N_\Lambda = N)$. Then, for any function $f \in L^2(\mu^{\eta}_{z,\Lambda})$ whose $2R+\delta-$support Δ satisfies $|\Delta| \leqslant |\Lambda|^{1-4\varepsilon}$, $\varepsilon \in (0,1)$, there exist $C = C(z_0, R, \varepsilon, \delta) > 0$ and $v = v(z_0, R, \varepsilon, \delta) > 0$ such that for all Λ such that $|\Lambda| \geqslant v$

$$\left|\nu^{\eta}_{\Lambda,N}(f) - \mu^{\eta}_{z,\Lambda}(f)\right| \leqslant C\,\mu^{\eta}_{z,\Lambda}(f,f)^{1/2}\,\frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|}.$$

If the function f has bounded uniform norm $||f||_\infty$ the result is the same as the discrete case: $\mu^{\eta}_{z,\Lambda}(f,f) \leqslant 4\,||f||^2_\infty \min\{z|\Delta|, 1\}$ (see ref. 10). The estimates we use to prove the above result are quite similar to those of the discrete case in ref. 10. We give here all the details for completeness and because to obtain the L^2-norm we had to refine some of them. The L^2-norm is more suitable in the continuous case because many observables, as for example the number of particles in a finite volume, have unbounded uniform norm.

Remark 2.2. As stated above we assume that $\mu^{\eta}_{z,\Lambda}(N_\Lambda) = N$, this means that the activity is conveniently chosen from the beginning as function of N, Λ and η. This has no consequences on the DLR property of the Gibbs measure since Λ and η are kept fixed once and for all.

Remark 2.3. It will be clear from the proof that, if $z \geqslant z_1 > 0$ uniformly in $|\Lambda|$, the condition $|\Delta| \leqslant |\Lambda|^{1-4\varepsilon}$ can be relaxed to $|\widetilde{\Delta}| = o(|\Lambda|)$ (see definition (3.3) of $\widetilde{\Delta}$).

Remark 2.4. To prove the result (1.2), Spohn needed more than the condition (CE). Indeed, he took $0 < z < 0.28\,\hat{z}_0(\beta,\phi)$ (see ref. 13).

In order to prove the theorem we need, as we stressed above, a mixing condition for the grand canonical Gibbs measure. One can show that the following strong mixing condition holds (see Corollary 2.4 in ref. 14 or Lemma 4 in ref. 13 for a proof):

Proposition 2.5. (Property (SMC)). Let z and β such that (CE) holds. There exist two constants $\alpha = \alpha(R, z, \beta)$ and $m = m(R, z, \beta)$ such that $\forall\ \Lambda,\ \Delta_f,\ \Delta_g \in \mathcal{B}(\mathbb{R}^d)$ such that $\Delta_f \subset \Lambda$, $\Delta_g \subset \Lambda$, $d(\Delta_f, \Delta_g) > 2R$ and $\min(|\bar{\Delta}_f^R|, |\bar{\Delta}_g^R|) \leqslant \exp(md(\Delta_f, \Delta_g))$, we have for $f \in \mathcal{F}_{\Delta_f}$ and $g \in \mathcal{F}_{\Delta_g}$

$$|\mu_{z,\Lambda}^{\eta}(f, g)| \leqslant \alpha\, \mu_{z,\Lambda}^{\eta}(|f|)\mu_{z,\Lambda}^{\eta}(|g|)\, e^{-md(\Delta_f, \Delta_g)},$$

where Δ_f and Δ_g are respectively the supports of f and g and $\bar{A}^R \doteq \{x \in \mathbb{R}^d\ \ d(x, A) \leqslant R\}$ for $A \subset \mathbb{R}^d$.

Remark 2.6. The constants α and m are respectively increasing and decreasing as functions of the activity z, $0 < z < \frac{1}{3}\hat{z}_0$, and for small z the constant m is proportional to $-\log z$ (see Lemma 4 in ref. 13).

This result has an immediate consequence, which will be useful for our purpose, see ref. 14 for the proof.

Corollary 2.7. If (CE) holds, there exist two constants $\widetilde{\alpha} = \widetilde{\alpha}(R, z, \beta)$ and $\widetilde{m} = \widetilde{m}(R, z, \beta)$ such that for all $\Lambda, \Delta_f \in \mathcal{B}(\mathbb{R}^d)$, $\Delta_f \subset \Lambda$,

$$|\mu_{z,\Lambda}^{\eta}(f) - \mu_{z,\Lambda}^{\omega}(f)| \leqslant \widetilde{\alpha}\, \mu_{z,\Lambda}^{\eta}(|f|)\, e^{-\widetilde{m}\, d(\Delta_f, \eta\triangle\omega)}$$

for all $\omega, \eta \in \Omega$, for all $f \in \mathcal{F}_{\Delta_f}$ such that $d(\Delta_f, \eta\triangle\omega) > 3R$, and $|\bar{\Delta}_f^R| \leqslant \exp[\widetilde{m}\,(d(\Delta_f, \eta\triangle\omega) - R)]$.

3. PROOF OF THEOREM 2.1

Through all the section c, c' will denote positive constants which do not depend on f, Λ, N and can change from line to line.
Fix $\Lambda \in \mathbb{F}_{2R+\delta}$ for some $\delta > 0$. To simplify the notations, we use

$$\begin{aligned}
\mu &\doteq \mu_{z,\Lambda}^{\eta}\,,\\
\nu &\doteq \nu_{N,\Lambda}^{\eta}\,,\\
\sigma &\doteq \mu(N_\Lambda, N_\Lambda)\,,\\
\bar{h} &\doteq h - \mu(h)\ \ \forall h \quad \mu\text{-}integrable\,.
\end{aligned}$$

Let χ_N be the indicator function of the event $\{N_\Lambda(\omega)=N\}$. Then we write that

$$\nu(f)-\mu(f)=\frac{\mu(f,\chi_N)}{\mu(\chi_N)}. \tag{3.1}$$

Using the Fourier transform, we can express χ_N as

$$\chi_N(\omega)=\frac{1}{2\pi\sigma}\int_{-\pi\sigma}^{\pi\sigma} dt\ e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda(\omega)}.$$

Therefore (3.1) becomes

$$\nu(f)-\mu(f)=\frac{\int_{-\pi\sigma}^{\pi\sigma} dt\,\mu\,(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda},\ f)}{\int_{-\pi\sigma}^{\pi\sigma} dt\,\mu\,(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})}. \tag{3.2}$$

The proof consists on a separated study of the numerator (Step 1) and the denominator of (3.2) (Step 2). We conclude the Proof of Theorem 2.1 in Step 3.

Step 1: Study of $\int_{-\pi\sigma}^{\pi\sigma} dt\ \ \mu(e^{i\frac{t}{\sigma}\bar{N}_\Lambda}, f)$.

We start by proving an upper bound for a rather special class of functions: the ones which have, roughly speaking, almost zero covariance with N_Λ. In Step 3, using the conservation law $N_\Lambda=N$, we shall extend the result to more general functions.

Let $l_0=2R+\delta$, and $\{Q_i\}_{i\in I_\Lambda}$ the partition of $\Lambda\in\mathbb{F}_{l_0}$. Given $\varepsilon\in(0,1)$ and $V\in\mathbb{F}_{l_0},\ \ V\subset\Lambda$, we define, for any positive large number M, the set

$$\widetilde{V}\doteq\begin{cases}\{\ Q_i\,,\ \ i\in I_\Lambda\ \mid\ d(Q_i,V)\leqslant M\log|\Lambda|\ \} & \text{if } z\geqslant|\Lambda|^{-\varepsilon},\\ \{\ Q_i\,,\ \ i\in I_\Lambda\ \mid\ d(Q_i,V)\leqslant M\ \} & \text{if } z\leqslant|\Lambda|^{-\varepsilon}.\end{cases} \tag{3.3}$$

Let g be a local function of l_0-support Δ and define $f\doteq g-\alpha_g N_{\Delta_1}$. The set $\Delta_1\subset\Lambda$, $\Delta_1\in\mathbb{F}_{l_0}$, has the following properties: *(i)* it can be written as $\Delta_1=\cup_{I'_{\Delta_1}}V_i$ where for all $i\in I'_{\Delta_1}$, there exist positive numerical constants c_i, k_i such that $|V_i|=c_i\,l_0^d$, $|\partial_{l_0}^- V_i|=k_i\,l_0^d$ and $c_i\,k_i^{-1}\geqslant 2z_0\,l_0^d\,e^{3z_0\,l_0^d}$; and *(ii)* there exists a positive numerical constant κ such that $\kappa^{-1}|\Delta|\leqslant|\Delta_1|\leqslant\kappa\,|\Delta|$. The l_0-support of f is $\Delta_f\doteq\Delta\cup\Delta_1$. To simplify the notation, we use $\widetilde{\Delta}_f\doteq\widetilde{\Delta}$. Let us define

$$\alpha_g\doteq\frac{\mu(g,N_{\widetilde{\Delta}})}{\mu(N_{\Delta_1},N_{\widetilde{\Delta}})}. \tag{3.4}$$

By Proposition 4.4, the above properties of Δ_1 and Remark 4.8, α_g is well defined and satisfies

$$|\alpha_g| \leqslant c\sqrt{\frac{\mu(g,g)}{z|\Delta_1|}}. \tag{3.5}$$

Then, the following lemma holds.

Lemma 3.1. There exists a positive constant $C_1 = C_1(z_0, R, \varepsilon, \delta)$ such that, if $f = g - \alpha_g N_{\Delta_1}$,

$$\int_{-\pi\sigma}^{\pi\sigma} dt \; \left|\mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda}, f)\right| \leqslant C_1\, \mu(g,g)^{\frac{1}{2}} \frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|}.$$

Step 2: Study of the denominator $\int_{-\pi\sigma}^{\pi\sigma} \mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\, dt$. We have the following Lemma:

Lemma 3.2. There exists a constant $C_2 = C_2(z_0, R, \delta)$ such that

$$\int_{-\pi\sigma}^{\pi\sigma} \mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\; dt \geqslant C_2.$$

Step 3: We conclude the Proof of Theorem 2.1.
By Lemmas 3.1 and 3.2, we have that for any function $f = g - \alpha_g N_{\Delta_1}$ with g an arbitrary function in $L^2(\mu)$, with compact l_0-support Δ and Δ_1 and α_g defined as in Step 1,

$$\left|\nu(f) - \mu(f)\right| \leqslant c\, \mu(g,g)^{\frac{1}{2}} \frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|}.$$

Thus,

$$\left|\nu(g) - \mu(g)\right| \leqslant c\, \mu(g,g)^{\frac{1}{2}} \frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|} + |\alpha_g| \left|\nu(N_{\Delta_1}) - \mu(N_{\Delta_1})\right|$$

using (3.5). To complete the proof, we need to prove the result for the special function N_{Δ_1}.
Let $\Delta_1 = \cup_{I'_{\Delta_1}} V_i$ verifying (i) and (ii) as before.
To simplify the notation, define $N_i \doteq N_{V_i}$. Also define $\Delta_{i,j} \doteq V_i \cup V_j$ and $f_{i,j} \doteq N_i - \alpha_{i,j} N_j$, $\alpha_{i,j} \doteq \mu(N_{\tilde{\Delta}_{i,j}}, N_i)/\mu(N_{\tilde{\Delta}_{i,j}}, N_j)$. By Proposition 4.4,

Inequality (4.16) (SMC) and the properties of the sets V_i there exists a positive constant $\alpha_0 = \alpha_0(R, z_0, \delta)$ such that $\alpha_{i,j} \geqslant \alpha_0$, $\forall i, j \in I'_{\Delta_1}$.
Define

$$R_{i,j} \doteq \nu(N_i) - \mu(N_i) - \alpha_{i,j}\left[\nu(N_j) - \mu(N_j)\right].$$

Applying Lemmas 3.1 and 3.2 to the function
$f_{i,j} \doteq N_i - \alpha_{i,j} N_j$ we have that

$$\sup_{i,j \in I'_{\Delta_1}} |R_{i,j}| \leqslant c\, z\, l_0^d \,\frac{\max\left\{1, \sqrt{z l_0^d}\right\}}{z|\Lambda|}.$$

By conservation law, $\sum_{i \in I'_{\Delta_1}} [\nu(N_i) - \mu(N_i)] = 0$. Then,

$$-\Big\{\sum_{i \in I'_{\Delta_1}} \alpha_{i,j}\Big\}\left[\nu(N_j) - \mu(N_j)\right] = \sum_{i \in I'_{\Delta_1}} R_{i,j}\,.$$

Thus,

$$|\nu(N_j) - \mu(N_j)| = \left|\frac{\sum_{i \in I'_{\Delta_1}} R_{i,j}}{\sum_{i \in I'_{\Delta_1}} \alpha_{i,j}}\right| \leqslant \frac{c}{\alpha_0}\, z\, l_0^d\, \frac{\max\left\{1, \sqrt{z l_0^d}\right\}}{z|\Lambda|}$$

and, by (3.5), we obtain

$$|\alpha_g|\,|\nu(N_{\Delta_1}) - \mu(N_{\Delta_1})| \leqslant c\, \mu(g,g)^{\frac{1}{2}}\, \frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|}\,.$$

The proof of the theorem is finished. ∎

3.1. Proof of Lemmas 3.1 and 3.2

3.1.1. Proof of Lemma 3.1

Define the sets

$$E \doteq \{\ x \in \widetilde{\Delta}^c \ \mid\ d(x, \widetilde{\Delta}) \leqslant R\ \},$$
$$F \doteq \Lambda \setminus (\widetilde{\Delta} \cup E)$$

and let

$$\begin{aligned} G_t(\eta) &\doteq \mu^{\eta}_{z,F}(e^{i\frac{t}{\sigma}\bar{N}_F}), \\ H_t(\eta) &\doteq e^{i\frac{t}{\sigma}\bar{N}_E(\omega)}, \\ K_t(\eta) &\doteq e^{-i\frac{t}{\sigma}\mu^{\eta}_{z,\widetilde{\Delta}}(\bar{N}_{\widetilde{\Delta}})}. \end{aligned}$$

By Markov property, we can write that

$$\mu(f, e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda}) = \mu(\bar{f}, e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda}) = \mu\left(H_t G_t K_t \mu_{z,\widetilde{\Delta}}(\bar{f}, e^{\mathrm{i}\frac{t}{\sigma}[N_{\widetilde{\Delta}} - \mu_{\widetilde{\Delta}}(N_{\widetilde{\Delta}})})\right) + R_t,$$

where $R_t = \mu\left(H_t G_t K_t \mu_{z,\widetilde{\Delta}}(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_{\widetilde{\Delta}}})\left[\mu_{z,\widetilde{\Delta}}(\bar{f}) - \mu(\bar{f})\right]\right)$.
By Proposition 4.6 and the fact that $|\widetilde{\Delta}| << |\Lambda|/2$, we have

$$||G_t||_\infty \leqslant e^{-ct^2}. \tag{3.6}$$

Thus we can prove the bound

$$\int_{-\pi\sigma}^{\pi\sigma} |R_t| \; dt \leqslant c\, \frac{\mu(g,g)^{1/2}}{|\Lambda|^2} \tag{3.7}$$

provided that the constant M appearing in the definition of $\widetilde{\Delta}$ verify $M > \max\{\frac{2}{m}, \frac{2}{\epsilon}\}$. Indeed, if $z \geqslant |\Lambda|^{-\varepsilon}$, we can write

$$\begin{aligned} |R_t| &\leqslant ||G_t||_\infty \mu\left(\sup_{\eta,\eta'\in\Omega}\left|\mu^{\eta}_{\widetilde{\Delta}}(\bar{f}) - \mu^{\eta'}_{\widetilde{\Delta}}(\bar{f})\right|\right) \\ &\leqslant ||G_t||_\infty \widetilde{\alpha}\;\; \mu(g,g)^{1/2} e^{-\widetilde{m} M \;\; \log|\Lambda|} \quad \text{by Corollary 2.7} \\ &\leqslant \widetilde{\alpha}\;\; e^{-ct^2} \frac{\mu(g,g)^{1/2}}{|\Lambda|^2} \quad \text{by (3.6)} \end{aligned}$$

provided that $M \geqslant 2/\widetilde{m}$. If $z \leqslant |\Lambda|^{-\varepsilon}$, the constant m in property (SMC) is proportional to $-\log\; z$ (see Lemma 4 in ref. 13) and then, using again Corollary 2.7 and (3.6),

$$|R_t| \leqslant ||G_t||_\infty\, \alpha\, \mu(g,g)^{1/2} e^{-mM} \leqslant \alpha\, e^{-c_1 t^2}\, \frac{\mu(g,g)^{1/2}}{|\Lambda|^{M\varepsilon}}$$

and (3.7) follows provided $M > 2/\epsilon$.

We thus have to study $\mu(|\mu_{z,\widetilde{\Delta}}(\bar{f}, e^{\mathrm{i}\frac{t}{\sigma}}\widehat{N}_{\widetilde{\Delta}})|)$, where $\widehat{N}_{\widetilde{\Delta}} = N_{\widetilde{\Delta}} - \mu_{z,\widetilde{\Delta}}(N_{\widetilde{\Delta}})$. We distinguish two cases: $z \geqslant |\Lambda|^{-\varepsilon}$ and $z \leqslant |\Lambda|^{-\varepsilon}$.

Case $z \geqslant |\Lambda|^{-\varepsilon}$. By a Taylor expansion up to the second order

$$\mu(|\mu_{\widetilde{\Delta}}(\bar{f}, e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_{\widetilde{\Delta}}})|) \leqslant \frac{|t|}{\sigma}\mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g}, N_{\widetilde{\Delta}}) - \alpha_g \mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1}, N_{\widetilde{\Delta}})\right|\right) \tag{3.8}$$

$$+\frac{t^2}{2\sigma^2}\mu\left(|\mu_{z,\widetilde{\Delta}}(\bar{g}, \widehat{N}^2_{\widetilde{\Delta}})| + |\alpha_g||\mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1}, \widehat{N}^2_{\widetilde{\Delta}})|\right) \tag{3.9}$$

$$+\delta_3,$$

where

$$\delta_3 \leqslant \frac{t^3}{\sigma^3}\mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g}, |\widehat{N}_{\widetilde{\Delta}}|^3)\right| + |\alpha_g|\left|\mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1}, |\widehat{N}_{\widetilde{\Delta}}|^3)\right|\right).$$

Let us start by (3.8), remembering the definition (3.4) of α_g, it can be bounded by

$$\mu\left\{\left|\frac{\mu_{z,\widetilde{\Delta}}(\bar{g}, N_{\widetilde{\Delta}})}{\mu(N_{\Delta_1}, N_{\widetilde{\Delta}})}\right|\left|\mu(N_{\Delta_1}, N_{\widetilde{\Delta}}) - \mu_{z,\widetilde{\Delta}}(N_{\Delta_1}, N_{\widetilde{\Delta}})\right|\right.$$
$$\left.+\left|\frac{\mu_{z,\widetilde{\Delta}}(N_{\Delta_1}, N_{\widetilde{\Delta}})}{\mu(N_{\Delta_1}, N_{\widetilde{\Delta}})}\right|\left|\mu_{z,\widetilde{\Delta}}(\bar{g}, N_{\widetilde{\Delta}}) - \mu(g, N_{\widetilde{\Delta}})\right|\right\}.$$

Let $\{Q_i\}_{i\in I_{\widetilde{\Delta}}}$ the $2R+\delta$-cubes of the partition of $\widetilde{\Delta}$. Then
$|\mu^{\sigma}_{z,\widetilde{\Delta}}(\bar{g}, N_{Q_i}) - \mu(\bar{g}, N_{Q_i})|$

$$\leqslant \frac{c}{|\Lambda|^{Mm/2}}\begin{cases}\mu^{\sigma}_{z,\widetilde{\Delta}}(|\bar{g}|)\mu^{\sigma}_{z,\widetilde{\Delta}}(N_{Q_i}) + \mu(|\bar{g}|)\mu(N_{Q_i}) & \text{if } d(\Delta, Q_i) \geqslant \frac{M}{2}\log|\Lambda|,\\ \mu^{\sigma}_{z,\widetilde{\Delta}}(|\bar{g}|N_{Q_i}) + \mu^{\sigma}_{z,\widetilde{\Delta}}(|\bar{g}|)\mu^{\sigma}_{z,\widetilde{\Delta}}(N_{Q_i}) & \text{if } d(\Delta, Q_i) \leqslant \frac{M}{2}\log|\Lambda|,\end{cases}$$

where we obtained the first inequality by the (SMC) property and the second one by Corollary 2.7. Thus,

$$\mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g}, N_{\widetilde{\Delta}}) - \alpha_g \mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta}, N_{\widetilde{\Delta}})\right|\right)$$
$$\leqslant \mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g}, N_{\widetilde{\Delta}}) - \mu(\bar{g}, N_{\widetilde{\Delta}})\right|\right) + |\alpha_g|\mu\left(\left|\mu_{z,\widetilde{\Delta}}(N_{\Delta_1}, N_{\widetilde{\Delta}}) - \mu(N_{\Delta_1}, N_{\widetilde{\Delta}})\right|\right)$$
$$\leqslant c\frac{|\widetilde{\Delta}|}{|\Lambda|^{\frac{Mm}{2}}}[\mu(|\bar{g}|) + \mu(g,g)^{\frac{1}{2}}(z|\Delta_1|)^{\frac{1}{2}}] \leqslant c\frac{\mu(g,g)^{\frac{1}{2}}}{|\Lambda|^{\frac{Mm}{2}-1}}. \tag{3.10}$$

Concerning (3.9), by points 5 and 6 of Proposition 4.4 and inequality (4.16), we have

$$\frac{t^2}{2\sigma^2}\mu\left(|\mu_{z,\widetilde{\Delta}}(\bar{g},\widehat{N}^2_{\widetilde{\Delta}})|+|\alpha_g||\mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1},\widehat{N}^2_{\widetilde{\Delta}})|\right)$$
$$\leqslant c\,\mu(g,g)^{1/2}\,t^2\,\frac{\max\{z|\Delta_1|,\sqrt{z|\Delta_1|}\}}{z|\Lambda|}\,. \tag{3.11}$$

For δ_3, we have by Schwarz inequality, points 2 and 3 of Proposition 4.4, (3.5) and (4.16)

$$\begin{aligned}\delta_3 \;&\leqslant\; \frac{|t|^3}{\sigma^3}\,\mu\left(\left\{\mu_{z,\widetilde{\Delta}}(|\bar{g}||\widehat{N}_{\widetilde{\Delta}}|^3)+\mu_{z,\widetilde{\Delta}}(|\bar{g}|)\mu_{z,\widetilde{\Delta}}(|\widehat{N}_{\widetilde{\Delta}}|^3)\right\}\right.\\ &\quad\left.+\,|\alpha_g|\left\{\mu_{z,\widetilde{\Delta}}(|\bar{N}_{\Delta_1}|\,|\widehat{N}_{\widetilde{\Delta}}|^3)+\mu_{z,\widetilde{\Delta}}(|\bar{N}_{\Delta_1}|)\mu_{z,\widetilde{\Delta}}(|\widehat{N}_{\widetilde{\Delta}}|^3)\right\}\right)\\ &\leqslant\; c\,\mu(g,g)^{1/2}\,\frac{\max\{z|\Delta_1|,(z|\Delta_1|)^3\}^{1/2}}{(z|\Lambda|)^{\frac{3}{2}}}\,(\log|\Lambda|)^2\\ &\leqslant\; c\,\mu(g,g)^{1/2}\,\frac{|\Delta_1|}{|\Lambda|},\end{aligned} \tag{3.12}$$

where we used that $z\geqslant|\Lambda|^{-\varepsilon}$ in the last inequality.
Combining (3.10), (3.11) and (3.12) and using the fact that $|\Delta_1|\leqslant\kappa\,|\Delta|$, we finally get

$$\mu(|\mu_{z,\widetilde{\Delta}}(f,e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_\Lambda})|)\leqslant c\,\mu(g,g)^{\frac{1}{2}}\,(|t|+t^2+|t|^3)\,\frac{\max\{z|\Delta|,\sqrt{z|\Delta|}\}}{z|\Lambda|}$$

provided that $M>4/m$. Using (3.6) and integrating in dt the result of lemma 3.1 follows for $z\geqslant|\Lambda|^{-\varepsilon}$.
Case $z\leqslant|\Lambda|^{-\varepsilon}$. By Taylor expansion up to the first order

$$\mu(|\mu_{z,\widetilde{\Delta}}(\bar{f},e^{i\frac{t}{\sigma}\widehat{N}_{\widetilde{\Delta}}})|)\leqslant\frac{|t|}{\sigma}\,\mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g},\widehat{N}_{\widetilde{\Delta}})-\alpha_g\,\mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1},\widehat{N}_{\widetilde{\Delta}})\right|\right)+\delta_2 \tag{3.13}$$

where

$$\delta_2\leqslant\frac{t^2}{2\sigma^2}\mu\left(\left|\mu_{z,\widetilde{\Delta}}(\bar{g},|\widehat{N}_{\widetilde{\Delta}}|^2)\right|+|\alpha_g|\,|\mu_{z,\widetilde{\Delta}}(\bar{N}_{\Delta_1},|\widehat{N}_{\widetilde{\Delta}}|^2)\right)\,.$$

The first-order term (3.13) can be estimated as in the case $z\geqslant|\Lambda|^{-\varepsilon}$ (see (3.10)). For δ_2, by Schwarz inequality, points 1 and 2 of Proposition 4.4, using (4.16) and that now $|\widetilde{\Delta}|$ is proportional to $|\Delta|$ we have

$$\delta_2\leqslant c\,t^2\,\mu(g,g)^{1/2}\,\frac{\max\{z|\Delta|,\sqrt{z|\Delta|}\}}{z|\Lambda|}\,.$$

Therefore

$$\mu(|\mu_{z,\widetilde{\Delta}}(f, e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_\Lambda})|) \leqslant c\,\mu(g,g)^{1/2}\,(|t|+t^2)\,\frac{\max\{z|\Delta|, \sqrt{z|\Delta|}\}}{z|\Lambda|}\,.$$

Using (3.6) and integrating in dt we obtain Lemma 3.1. ∎

3.1.2. Proof of Lemma 3.2

Fix a real number $M_1 >> 1$. We distinguish two cases. *Case 1:* $\sigma \geqslant M_1^{d+5}$. We write that

$$\int_{-\pi\sigma}^{\pi\sigma} \mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\ dt = I_1 + I_2,$$

where

$$I_1 = \int_{-M_1}^{M_1} \mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\ dt,$$

and

$$I_2 = \int_{M_1 \leqslant |t| \leqslant \pi\sigma} \mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\ dt\,.$$

By Proposition 4.6 and Remark 4.8, we have that $|I_2| \leqslant e^{-D M_1^2}$, $D = D(R, z_0, \delta)$. So we have to study I_1.

Let V a $l_0 = 2R + \delta$-regular subset of Λ with the properties: (i) it can be written as the union of l_0-regular subsets V_j, i.e., $V = \cup_{j\in I_V} V_i$, and (ii) $|\Lambda \setminus V|/|V| \leqslant M_1^{-1}$; $d(V\setminus V_j, V_j) \geqslant 2R+\delta$; $\mathrm{diam}|V_j| \leqslant M_1 l_0$ and $|\Lambda|/|V| \geqslant 1/2$. To simplify the notation let $\widehat{N}_V \doteq N_V - \mu_{z,V}(N_V)$. We write

$$\begin{aligned}
\mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda}) &= \mu\left(e^{\mathrm{i}\frac{t}{\sigma}\{\bar{N}_{\Lambda\setminus V} + \mu_{z,V}(\bar{N}_V)\}}\mu_{z,V}(e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_V})\right)\\
&= \mu\left(\mu_{z,V}(e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_V})\right)\\
&\quad + \mu\left(\mu_{z,V}(e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_V})\left(e^{\mathrm{i}\frac{t}{\sigma}\{\bar{N}_{\Lambda\setminus V} + \mu_{z,V}(\bar{N}_V)\}} - 1\right)\right) \qquad (3.14)
\end{aligned}$$

Since the measure $\mu_{z,V}$ factorize, we have that

$$F_V(\frac{t}{\sigma}) \doteq \mu_{z,V}(e^{\mathrm{i}\frac{t}{\sigma}\widehat{N}_V}) = \Pi_{j\in I_V}\mu_{z,V_j}(e^{\mathrm{i}\frac{t}{\sigma}[N_{V_j} - \mu_{z,V_j}(N_{V_j})]}) = \prod_{j\in I_V} F_{V_j}(\frac{t}{\sigma})\,.$$

By Taylor expansion up to the second order we get

$$F_{V_j}(\frac{t}{\sigma}) = 1 - \frac{t^2}{2\sigma^2}\mu_{z,V_j}(N_{V_j}, N_{V_j}) + R_j,$$

where

$$|R_j| \leqslant \frac{|t|^3}{\sigma^3}\mu_{z,V_j}(|N_{V_j} - \mu_{z,V_j}(N_{V_j})|^3) \leqslant 2\,\frac{|t|^3}{\sigma^3}\,(M_1 l_0)^d\,\mu_{z,V_j}(N_{V_j}, N_{V_j})$$

(using the (SMC) property).
By point 1 of Proposition 4.4 and (4.8) we have

$$\frac{\sum_{j\in I_V}\mu_{z,V_j}(N_{V_j}, N_{V_j})}{\sigma^2} \leqslant \frac{A}{D_1}\,\frac{\sum_j z|V_j|}{z|\Lambda|} \leqslant \frac{A}{D_1}\,\frac{|V|}{|\Lambda|} \leqslant 2\frac{A}{D_1},$$

where we used $|\Lambda|/|V| \geqslant 1/2$ in the last inequality.
Therefore, for any $t \in [-M_1, M_1]$,

$$\sum_{j\in I_V}|R_j| \leqslant D_2\frac{M_1^{3+d}}{\sigma} \leqslant c\,M_1^{-2} \qquad \text{because } \sigma \geqslant M_1^{d+5}.$$

We then deduce that for M_1 large enough,

$$\left|\mu\left(F_V(\frac{t}{\sigma})\right) - \mu\left(\Pi_j\left[1 - \frac{t^2}{2\sigma^2}\mu_{z,V_j}(N_{V_j}, N_{V_j})\right]\right)\right| \leqslant 3\sum_j |R_j| \leqslant 3\,c\,M_1^{-2}.$$

Thus, for M_1 large enough, we obtain that

$$\begin{aligned}\int_{-M_1}^{M_1}\mu\left(F_V(\frac{t}{\sigma})\right)dt &\geqslant \int_{-M_1}^{M_1}\mu\left(\Pi_j\left[1 - \frac{t^2}{2\sigma^2}\mu_{z,V_j}(N_{V_j}, N_{V_j})\right]\right)\,dt - 3\,c\,M_1^{-1}\\ &\geqslant \int_{-M_1}^{M_1} e^{-c' t^2}\,dt - 3\,c\,M_1^{-1} \geqslant c\ . \end{aligned} \tag{3.15}$$

Notice that we can estimate $1 - t^2/2\sigma^2\mu_{z,V_j}(N_{V_j}, N_{V_j})$ in terms of a negative exponential because we have the following upper bound (using point 2 of Proposition 4.4):

$$\frac{t^2}{2\sigma^2}\mu_{z,V_j}(N_{V_j}, N_{V_j}) \leqslant \frac{A}{2}\,M_1^{2-2(d+5)} z|V_j| \leqslant \frac{zA}{2}\,l_0^d\,M_1^{-(d+8)}.$$

Up to this point, in order to have the result, we need to bound the integral of the second term of (3.14). By Taylor expansion up to the first order and using Proposition 4.6 and (4.8), we get

$$
\begin{aligned}
|\mu(F_V(\frac{t}{\sigma})\,(e^{\mathrm{i}\frac{t}{\sigma}[\bar N_{\Lambda\setminus V}+\mu_{z,V}(\bar N_V)]}-1))| &\leqslant \frac{|t|}{\sigma}\,\mu(|F_V(\frac{t}{\sigma})|\,[|\bar N_{\Lambda\setminus V}|+|\mu_{z,V}(\bar N_V)|])\\
&\leqslant \frac{|t|}{\sigma}\,e^{-ct^2}\,[\mu(|\bar N_{\Lambda\setminus V}|)+\mu(|\mu_{z,V}(\bar N_V)|)]\,.
\end{aligned}
\tag{3.16}
$$

By Schwarz inequality and point 1 of Proposition 4.4, we have

$$
\frac{|t|}{\sigma}\mu(|\bar N_{\Lambda\setminus V}|)\leqslant A\frac{|t|}{\sigma}(z|\Lambda\setminus V|)^{1/2}\leqslant \frac{A}{D_1}\,|t|M_1^{-1/2}
$$

because $|\Lambda\setminus V|/|V|\leqslant M_1^{-1}$ and by (4.16) $\sigma^2\geqslant D_1\,z|V|$.
We now bound the second term on the right hand side of inequality (3.16). By Schwarz inequality we have

$$
\mu(|\mu_{z,V}(\bar N_V)|)\leqslant \mu\left(\mu_{z,V}(N_V),\mu_{z,V}(N_V)\right)^{1/2},
$$

so that we can use Poincaré inequality given by Theorem 4.1 for the function $f(\eta)=\mu^{\eta}_{z,V}(N_V)$. As

$$
D_x^+f=D_x^+\mu^{\eta}_{z,V}(N_V)=\begin{cases}0 & \text{if } x\in V,\\ \mu^{\eta\cup x}_{z,V}(N_V)-\mu^{\eta}_{z,V}(N_V) & \text{if } x\in\Lambda\setminus V\end{cases}
$$

and by Corollary 2.7 and $V\in\mathbb{F}_{l_0}$ we have

$$
|\mu^{\eta\cup x}_{z,V}(N_V)-\mu^{\eta}_{z,V}(N_V)|\leqslant c\,z\,l_0^d
$$

so that by Poincaré inequality, i.e., Theorem 4.1

$$
\mu\left(\mu_{z,V}(N_V),\mu_{z,V}(N_V)\right)\leqslant c\,\mathcal{E}^{\eta}_{z,\Lambda}(\mu_{z,V}(N_V),\mu_{z,V}(N_V))\leqslant c\,z\,|\Lambda\setminus V|\ .
$$

Finally, using also (4.16), we have

$$
\frac{|t|}{\sigma}\mu(|\mu_{z,V}(\bar N_V)|)\leqslant \frac{c}{D_1}\,|t|M_1^{-1/2}\,.
$$

We thus get

$$\left|\mu\left(\mu_{z,V}(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_V})\left(e^{\mathrm{i}\frac{t}{\sigma}\{\bar{N}_{\Lambda\setminus V}+\mu_{z,V}(\bar{N}_V)\}}\right)-1\right)\right|\leqslant c'\,|t|\,M_1^{-1/2}\,e^{-ct^2}, \quad (3.17)$$

combining (3.15) and (3.17), we obtain

$$\int_{-M_1}^{M_1}\mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\;dt\geqslant c-c'M_1^{-\frac{1}{2}}\int_{-M_1}^{M_1}|t|e^{-ct^2}\geqslant c-c'M_1^{-1/2}\geqslant c$$

for M_1 large enough.

Case 2: $\sigma\leqslant M_1^{d+5}$. Due to point 1 of Proposition 4.4 if $|\Lambda|^{1/2}$ is large compared to M_1^{d+5} this corresponds to extremely low particles density: by (4.8), $M_1^{d+5}\geqslant\sigma\geqslant D_1'\sqrt{N}$, with $D_1'=D_1'(z_0,R,\delta)$. We have

$$\int_{-\pi\sigma}^{\pi\sigma}\mu(e^{\mathrm{i}\frac{t}{\sigma}\bar{N}_\Lambda})\;dt=2\pi\sigma\mu(N_\Lambda=N)\geqslant c\,\mu(N_\Lambda=N)\,.$$

Therefore, we have to estimate $\mu(N_\Lambda=N)$. As $N\leqslant M_1^{2d+10}/D_1'$ we can find subsets $\{V_i\}_{i=1}^N$ of Λ such that $d(V_i,V_j)>2R,\ i\neq j,\ |V_i|\geqslant a\,|\Lambda|\,D_1'/M_1^{2d+10}$, $0<a<1$ a suitable numerical constant. Therefore

$$\begin{aligned}\mu(N_\Lambda=N) &= \frac{1}{Z_{z,\Lambda}^\eta}\frac{z^N}{N!}\int_{\Lambda^N}e^{-\beta H_\Lambda^\eta(x)}\;dx\\ &\geqslant \frac{1}{Z_{z,\Lambda}^\eta}\frac{z^N}{N!}\prod_{i=1}^N\int_{V_i}dx\\ &\geqslant \frac{1}{Z_{z,\Lambda}^\eta}\frac{z^N|\Lambda|^N}{N!}\left(a\,\frac{D_1'}{M_1^{2d+10}}\right)^N\\ &\geqslant \left(a\,\frac{D_1'}{M_1^{2d+10}}\,A^{-1}\right)^N\frac{N^N}{N!\,Z_{z,\Lambda}^\eta}\geqslant c\end{aligned}$$

where we used point 1 of Proposition 4.4. ∎

4. TECHNICAL RESULTS

Through all the Section c will denote a positive constant which does not depend on f,Λ,N and can change from line to line.

We first recall the Poincaré inequality proved in ref. 14 for a continuous gas. Then, we give bounds on various kind of covariance for the finite volume grand canonical Gibbs measure when (CE) is assumed. Finally, we give a Gaussian upper bound on the characteristic function of the variable N_Λ.

4.1. The Poincaré Inequality

For a given function f on Ω, we let

$$D_x^+ f(\omega) := f(\omega \cup x) - f(\omega), \quad \omega \in \Omega, \; x \in \mathbb{R}^d .$$

We define

$$\mathcal{E}^\eta_{z,\Lambda}(f) := z \int_\Lambda dx \, \mu^\eta_{z,\Lambda}(e^{-\beta D_x^+ H^\eta_\Lambda(x)} \, |D_x^+ f|^2)$$

and $\mathcal{D}(\mathcal{E}^\eta_{z,\Lambda}) := \{f : \mathcal{E}^\eta_{z,\Lambda}(f) < \infty\}$.
The following theorem is proven in ref. 14 (Theorem 2).

Theorem 4.1. Assume (CE). There exists a finite constant $G = G(R, z, \beta)$ such that for all $\eta \in \Omega$, $\Lambda \subset \mathbb{R}^d$ the following inequality holds

$$\mu^\eta_{z,\Lambda}(f, f) \leqslant G \, \mathcal{E}^\eta_{z,\Lambda}(f) .$$

Remark 4.2. In fact, $\mathcal{E}^\eta_{z,\Lambda}$ is the Dirichlet form of a Glauber type dynamics of a continuous gas (see ref. 14).

Remark 4.3. The Poincaré constant is such that $G(R, z, \beta) \leqslant G(R, z_0, \beta)$, (see the proof of Theorem 2 in ref. 14, see also Corollary 5.1 in ref. 15).

4.2. Bounds on Covariance

The following proposition has been widely used in the proof of the main result.

Proposition 4.4. Assume (CE). Fix $\delta > 0$ and take $\Lambda \in \mathbb{F}_{2R+\delta}$. Then, for all $\eta \in \Omega$ and any function $f \in L^2(\mu^\eta_{z,\Lambda})$ with $2R + \delta$-support $\Delta \subset \Lambda$, there exists a positive constant $A = A(R, z_0, \delta)$ such that the following statements hold

1. $A^{-1} z|\Lambda| \leqslant \mu(N_\Lambda) \leqslant Az|\Lambda|$.
2. $\mu^\eta_{z,\Lambda}(\bar{N}^2_\Lambda) \leqslant A\, z|\Lambda|$.

3. $\mu^{\eta}_{z,\Lambda}(\bar{N}^4_{\Lambda}) \leqslant A \max\{z|\Lambda|, (z|\Lambda|)^2\}$.
4. $\mu^{\eta}_{z,\Lambda}(\bar{N}^6_{\Lambda}) \leqslant A \max\{z|\Lambda|, (z|\Lambda|)^3\}$.
5. $\left|\mu^{\eta}_{z,\Lambda}(f, N_{\Lambda})\right| \leqslant A\, \mu^{\eta}_{z,\Lambda}(f,f)^{1/2} \sqrt{z|\Delta|}$.
6. $\left|\mu^{\eta}_{z,\Lambda}(f, \bar{N}^2_{\Lambda})\right| \leqslant A\, \mu^{\eta}_{z,\Lambda}(f,f)^{1/2} \max\{z|\Delta|, \sqrt{z|\Delta|}\}$.
7. $\left|\mu^{\eta}_{z,\Lambda}(N_{\Delta}, \bar{N}^2_{\Lambda})\right| \leqslant A\, z|\Delta|$.

where $\bar{N}_{\Lambda} \doteq N_{\Lambda} - \mu^{\eta}_{z,\Lambda}(N_{\Lambda})$.

To prove Proposition 4.4, we need the following key estimates:

Lemma 4.5. Fix $\delta > 0$. Let $Q = Q_l(x)$, $x \in \mathbb{R}^d$. Then, for all $\eta \in \Omega$,

$$\mu^{\eta}_{z,Q}(N_Q) \leqslant e^{z|Q|}\, z\,|Q|\,, \tag{4.1}$$
$$\mu^{\eta}_{z,Q}(N^2_Q) \leqslant e^{z|Q|} \max\{z|Q|, (z|Q|)^2\}\,, \tag{4.2}$$
$$\mu^{\eta}_{z,Q}(N^4_Q) \leqslant e^{z|Q|} \max\{z|Q|, (z|Q|)^4\}\,, \tag{4.3}$$
$$\mu^{\eta}_{z,Q}(N^6_Q) \leqslant e^{z|Q|} \max\{z|Q|, (z|Q|)^3\}\,. \tag{4.4}$$

Furthermore if $l \geqslant 2R + \delta$, there exists a numerical positive constant c such that

$$\mu^{\eta}_{z,Q}(N_Q) \geqslant c^{-1}\, e^{-z|Q|}\, z\,|Q|\,. \tag{4.5}$$

Proof of Lemma 4.5. By definition, we have

$$Z^{\eta}_{z,Q} = \sum_{k=0}^{+\infty} \frac{z^k}{k!} \int_{Q^k} dx\, e^{-\beta H^{\eta}_Q(x)}\,.$$

Since the pair potential $\phi \geqslant 0$, we easily deduce the following bound

$$1 \leqslant Z^{\eta}_{z,Q} \leqslant e^{z|Q|}. \tag{4.6}$$

Furthermore,

$$\mu^{\eta}_{z,Q}(N_Q) = \sum_{k=1}^{+\infty} k \mu^{\eta}_{z,Q}(N_Q = k) = \frac{1}{Z^{\eta}_{z,Q}} \sum_{k=1}^{+\infty} \frac{k z^k}{k!} \int_{Q^k} dx\, e^{-\beta H^{\eta}_Q(x)}\,.$$

Using that ϕ is positive and (4.6), we get

$$\mu^{\eta}_{z,Q}(N_Q) \leqslant \frac{z|Q|e^{z|Q|}}{Z^{\eta}_{z,Q}} \leqslant z|Q|e^{z|Q|}.$$

Moreover,

$$\mu^{\eta}_{z,Q}(N_Q) \geqslant \mu^{\eta}_{z,Q}(N_Q = 1) = \frac{1}{Z^{\eta}_{z,Q}} z \int_Q dx \; e^{-\beta H^{\eta}_Q(x)}.$$

As l is strictly bigger than $2R$, one can construct a cube $\widetilde{Q} \subset Q$ such that $\forall x \in \widetilde{Q}$, $H^{\eta}_Q(x) = 0$ (because there is only one particle in the cube Q). Therefore, $\mu^{\eta}_{z,Q}(N_Q) \geqslant z|\widetilde{Q}|/Z^{\eta}_{z,Q}$. Thus

$$z\,|\widetilde{Q}|\,e^{-z|Q|} \leqslant \mu^{\eta}_{z,Q}(N_Q) \leqslant z\,|Q|e^{z|Q|}. \tag{4.7}$$

As there exists a numerical positive constant c such that $|\widetilde{Q}| \geqslant c^{-1}|Q|$ the bound (4.5) follows.
Inequality (4.1) can be obtained by

$$\begin{aligned}\mu^{\eta}_{z,Q}(N_Q^2) = \sum_{k=1}^{+\infty} k^2 \mu^{\eta}_{z,Q}(N_Q = k) &\leqslant \frac{1}{Z^{\eta}_{z,Q}} \sum_{k=1}^{+\infty} \frac{k^2 z^k |Q|^k}{k!} \\ &= \frac{1}{Z^{\eta}_{z,Q}} z|Q|e^{z|Q|}(1 + z|Q|) \\ &\leqslant z|Q|e^{z|Q|}(1 + z|Q|),\end{aligned}$$

where we used again that ϕ is positive and (4.6).
The proofs of (4.3) and of (4.4) are analogous. ▮

Proof of Proposition (4.4). To simplify the notations, we write $\mu \doteq \mu^{\eta}_{z,\Lambda}$, $\bar{f} \doteq f - \mu(f)$ for any observable f, and $l_0 = 2R + \delta$. As $\Lambda \in \mathbb{F}_{l_0}$ let $\{Q_i\}_{i \in I_\Lambda}$ be the partition of cubes of side l_0 of Λ. We also use $|Q_i| \doteq |Q| = l_0^d \;\forall\; i \in I_\Lambda$ and define $B = B(R, z_0, \delta) \doteq e^{z_0|Q|}$.

1. Using the partition of Λ, (4.1) and (4.5), the result follows.
2. Using the partition of Λ we can write

$$\mu(\bar{N}_\Lambda^2) \leqslant \sum_{i,j \in I_\Lambda; i \neq j} \mu(N_{Q_i}, N_{Q_j}) + \sum_{i \in I_\Lambda} \mu(N_{Q_i}^2)$$

and by (SMC), (4.1) and (4.1), we obtain that

$$\mu(\bar{N}_\Lambda^2) \leqslant \alpha \sum_{i,j\in I_\Lambda; i\neq j} \mu(N_{Q_i})\mu(N_{Q_j})e^{-md(Q_i,Q_j)} + \sum_{i\in I_\Lambda} \mu(N_{Q_i}^2)$$
$$\leqslant \alpha B^2\ z^2|Q|^2 \frac{|\Lambda|}{|Q|} + B\ \frac{|\Lambda|}{|Q|} \max(z|Q|, z^2|Q|^2)\,.$$

The result follows.

3. We can write

$$\mu(\bar{N}_\Lambda^4) \leqslant \sum_{i_1,i_2,i_3,i_4\in I_\Lambda} \left|\mu(\bar{N}_{Q_{i_1}}\bar{N}_{Q_{i_2}}\bar{N}_{Q_{i_3}}\bar{N}_{Q_{i_4}})\right|. \tag{4.8}$$

So, using Lemma 4.5 and the (SMC), we can bound

$$\left|\mu(\bar{N}_{Q_{i_1}}\bar{N}_{Q_{i_2}}\bar{N}_{Q_{i_3}}\bar{N}_{Q_{i_4}})\right|$$

by

(i) $B\max\{z|Q|, (z|Q|)^4\}$ if $i_1=i_2=i_3=i_4$;

(ii) $B\max\{z|Q|, (z|Q|)^2\}^2$ if $i_1=i_2\neq i_3=i_4$;

(iii) $B\max\{z|Q|, (z|Q|)^2\}\left[e^{-md(Q_{i_1},Q_{i_4}\cup Q_{i_2})}+e^{-md(Q_{i_2},Q_{i_4})}\right]$ if $i_1=i_2=i$, $i_3=j$ and $i_4=k$ with $i\neq j$, $i\neq k$;

(iv) $B\max\{z|Q|, (z|Q|)^3\}z|Q|e^{-md(Q_i,Q_j)}$ if $i_1=i_2=i_3=i$, $i_4=j$ with $i\neq j$;

(v) $B\max\{(z|Q|)^2, (z\,|Q|)^4\}e^{-md(i_1,i_2,i_3,i_4)}$ if $d(i_1,i_2,i_3,i_4)>8R$ and all cubes are different, where $d(i_1,i_2,i_3,i_4)=\sum_{k=1}^4 d(Q_{i_k}, \cup_{j\neq k} Q_{i_j})$;

and the result follows.

4. The Proof is the same as point 3.

5. For $j=1,2,\ldots$, let $\Delta_j \doteq \{Q_i,\ i\in I_\Lambda \mid jl_0 \leqslant d(Q_i,\Delta) \leqslant (j+1)l_0\}$. Call $N_j \doteq \sum_{i\in I_{\Delta_j}} N_{Q_i}$. We write

$$\left|\mu(f,N_\Lambda)\right| \leqslant 2\sum_{j\geqslant 1}\left|\mu(f,N_j)\right| + \left|\mu(f,N_\Delta)\right|. \tag{4.9}$$

By (SMC), (4.1) and Schwarz inequality, we have

$$\left|\mu(f,N_j)\right| \leqslant \alpha\,\mu(|\bar{f}|)\mu(N_j)\,e^{-mjl} \leqslant \alpha\,B\,\mu(f,f)^{1/2}\ z\,|\Delta_j|\,e^{-mjl},$$

and also

$$\begin{aligned}\left|\mu(f, N_j)\right| = \left|\mu(\bar{f}, \bar{N}_j)\right| &\leqslant \alpha\, \mu(|\bar{f}|)(\mu(N_j, N_j))^{1/2} e^{-mjl} \\ &\leqslant \alpha \sqrt{B}\, \mu(f, f)^{1/2} \sqrt{z|\Delta_j|}\, e^{-mjl}. \end{aligned} \tag{4.10}$$

Therefore, as there exists a numerical constant c such that $|\Delta_j| \leqslant c\,(j+1)^d |\Delta|$, we have

$$\left|\mu(f, N_j)\right| \leqslant c'\, \mu(f, f)^{1/2} \min\{z|\Delta|, \sqrt{z|\Delta|}\}\, e^{-mjl}\,. \tag{4.11}$$

with $c' = \alpha \min\{B, \sqrt{B}\}$.
On other hand, by Schwarz inequality,

$$\left|\mu(f, N_0)\right| \leqslant \sqrt{B}\, \mu(f, f)^{1/2} \sqrt{z|\Delta|}\,. \tag{4.12}$$

Finally, by (4.9), (4.11) and (4.12), we get the result.

6. Let the Δ_j be defined as in point 5. We write that

$$\left|\mu(f, \bar{N}_\Lambda^2)\right| = \left|\mu(\bar{f}\bar{N}_\Lambda^2)\right| \leqslant 2 \sum_{j \leqslant k} \left|\mu(\bar{f}\bar{N}_j \bar{N}_k)\right|. \tag{4.13}$$

Using (SMC), Schwarz inequality and 2 -, we find
$\left|\mu(\bar{f}\bar{N}_j\bar{N}_k)\right| \leqslant$

$$c\,\mu(f, f)^{\frac{1}{2}} \begin{cases} \sqrt{z|\Delta_j|} \min\{z|\Delta_k| \sqrt{z|\Delta_k|}\}\, e^{-m(k-j)l} & \text{if } j \geqslant 1 \text{ and } k \geqslant 2j\,, \\ z\sqrt{|\Delta_j|}\sqrt{|\Delta_k|}\, e^{-mlj} & \text{if } j \geqslant 1 \text{ and } k < 2j\,, \\ \sqrt{z|\Delta|}\, \min\{z|\Delta_k|, \sqrt{z|\Delta_k|}\} e^{-mkl} & \text{if } j = 0 \text{ and } k \geqslant 1\,, \\ \max\{z|\Delta|, \sqrt{z|\Delta|}\} & \text{if } j = 0 \text{ and } k = 0\,. \end{cases}$$

Using again the bound $|\Delta_k| \leqslant c\,(k+1)^d |\Delta|$, the result comes from (4.13).

7. As above one has

$$\left|\mu(N_\Delta, \bar{N}_\Lambda^2)\right| \leqslant \sum_{i \in I_\Delta} \left|\mu(\bar{N}_{Q_i} \bar{N}_\Lambda^2)\right|. \tag{4.14}$$

Applying 6. to each term in the sum the result follows. ∎

4.3. Gaussian Upper Bound

Here, we prove a Gaussian upper bound for the characteristic function of the random variable N_Λ.

Proposition 4.6. Let $\Lambda \in \mathbb{F}_{2R+\delta}$, $\delta > 0$. For any $t \in [-\pi, \pi]$, there exists a positive numerical constant c such that

$$\left| \mu^{\eta}_{z,\Lambda}(e^{it[N_\Lambda - \mu^{\eta}_{z,\Lambda}(N_\Lambda)]}) \right| \leqslant \exp\{-c\, z\, e^{-z(2R+\delta)^d} |\Lambda| t^2\}. \tag{4.15}$$

Remark 4.7. Estimate (4.15) is called Gaussian upper bound. As in the discrete case (see refs. 8 and 10), it is proven by assuming only finite range and bounded interaction.

Remark 4.8. From inequality (4.15), it follows that

$$\mu^{\eta}_{z,\Lambda}(N_\Lambda, N_\Lambda) \geqslant D_1\, z\, |\Lambda|, \tag{4.16}$$

where $D_1 = 2c\, e^{-z(2R+\delta)^d}$.

Proof. As $\Lambda \in \mathbb{F}_{l_0}$, $l_0 = 2R + \delta$, let $\{Q_i\}_{i \in I_\Lambda}$ be the collection of cubes of the partition of Λ. To simplify, we write $\mu \doteq \mu^{\eta}_{z,\Lambda}$, $|Q| \doteq |Q_i|$, $i \in I_\Lambda$.

Let $\Lambda_R \doteq \{Q_i \ i \in I_\Lambda \mid d(Q_i, Q_j) > 2R\}$. There exists a numerical positive constant $c_1 > 1$ such that $|\Lambda_R| \geqslant (1/c_1)|\Lambda|$.
We write

$$\begin{aligned} \left| \mu(e^{itN_\Lambda}) \right| &= \left| \mu(e^{it\sum_{j\in I_{\Lambda_R}} N_{Q_j}} e^{itN_{\Lambda\setminus\Lambda_R}}) \right| \\ &= \left| \mu\left(\Pi_{j\in I_{\Lambda_R}} \mu_{z,Q_j}(e^{itN_{Q_j}}) \mu_{\Lambda\setminus\Lambda_R}(e^{itN_{\Lambda\setminus\Lambda_R}})\right) \right| \\ &\leqslant \left[\sup_{j,\omega} \left| \mu^{\omega}_{z,Q_j}(e^{itN_{Q_j}}) \right| \right]^n, \end{aligned}$$

where we used DLR Eq. (2.1) and $n = |I_{\Lambda_R}|$.
Set $g_j(t) = \left|\mu^{\omega}_{z,Q_j}(e^{itN_{Q_j}})\right|$. Using that $\forall\ x \geqslant 0$, $x \leqslant e^{(x^2-1/2)}$, we have

$$g_j(t) \leqslant \exp\left[\frac{1}{2}\left(g_j^2(t) - 1\right)\right]. \tag{4.17}$$

By an explicit computation, we obtain

$$g_j^2(t) - 1 = -\left[Var^{\omega}_{Q_j}(\cos(tN_{Q_j})) + Var^{\omega}_{Q_j}(\sin(tN_{Q_j}))\right],$$

where $Var^{\omega}_{Q_j}$ stands for the variance w.r.t. μ^{ω}_{z,Q_j}.

But, as $t\in[-\pi,\pi]$,

$$
\begin{aligned}
&Var^{\omega}_{Q_j}(\cos(tN_{Q_j}))+Var^{\omega}_{Q_j}(\sin(tN_{Q_j}))\\
&\quad= \frac{1}{2}\sum_{k,n}\Big\{\mu^{\omega}_{z,Q_j}(N_{Q_j}=k)\mu^{\omega}_{z,Q_j}(N_{Q_j}=n)\\
&\qquad\times\Big[(\cos(tk)-\cos(tn))^2+(\sin(tk)-\sin(tn))^2\Big]\Big\}\\
&\quad\geqslant \frac{1}{2}\mu^{\omega}_{z,Q_j}(N_{Q_j}=0)\mu^{\omega}_{z,Q_j}(N_{Q_j}=1)\Big[(\cos t-1)^2+\sin^2 t\Big]\\
&\quad\geqslant \frac{1}{2}\mu^{\omega}_{z,Q_j}(N_{Q_j}=0)\mu^{\omega}_{z,Q_j}(N_{Q_j}=1)\,c\,t^2.
\end{aligned}
$$

By (4.6), $\mu^{\omega}_{z,Q_j}(N_{Q_j}=0)=1/Z^{\omega}_{z,Q_j}\geqslant e^{-z|Q_j|}$.

On the other hand, proceeding as in the Proof of Lemma 4.5, we have

$$\mu^{\omega}_{z,Q_j}(N_{Q_j}=1)\geqslant\frac{c\,z|Q|}{e^{z|Q|}}.$$

Thus

$$g_j^2(t)-1\leqslant-\frac{c}{2}\,e^{-2z|Q|}\,z|Q|t^2$$

and then by (4.17),

$$g_j(t)\leqslant\exp\{-\frac{c}{4}e^{-2z|Q|}\,z|Q|t^2\}$$

so

$$\left|\mu(e^{\mathrm{i}tN_\Lambda})\right|\leqslant\exp\{-\frac{c_2\,c_3}{4}\,e^{-2z|Q|}\,z|\Lambda_R|\,t^2\}.$$

The result follows using that $|\Lambda_R|\geqslant c_1^{-1}|\Lambda|$. ∎

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